Determining the photon density from background radiation

Wiger van Houten
<crown.of.locusts@gmail.com>

What's the number of photons per unit volume excited in a cavity at temperature T? From Planck's law we know the mean number of photons excited in a certain mode (set of wavevector $\vec{k}$, and polarization λ) at a certain temperature. As the universe is basically a black box (no light escapes), this is a good starting point [1]:

$$\langle n(\omega) \rangle = \frac{1}{e^{(\hbar\omega/kT)} - 1}. \tag{1}$$

We also know the density of states [2]

$$\rho(\omega) d\omega = \frac{\omega^2 d\omega}{\pi^2 c^3}. \tag{2}$$

So we know the amount of photons excited in a mode, and the number of modes per unit volume. To find the number of photons from the background radiation per litre, we simply have to multiply both formulas and integrate:

$$\int_0^\infty \langle n(\omega) \rangle \rho(\omega) d\omega = \frac{1}{\pi^2 c^3} \left(\frac{kT}{\hbar} \right)^3 \int_0^\infty \frac{x^2}{e^x - 1} dx, \tag{3}$$

where $x = \frac{\hbar\omega}{kT}$ was substituted. The integral isn't one you want to solve yourselve (it's the Riemann Zeta-function), but with some help from matlab we find

$$\int_0^\infty \frac{x^2}{e^x - 1} = 2\zeta(3) \approx 2.4041. \tag{4}$$

Now all we need to do is fill in the constants. For the temperature T we take 2.728K: this is the temperature which fits the black body spectrum as observed by measurements (cosmic background radiation). Filling in the values for the constants: $c = 2.998 \cdot 10^8$, $k = 1.381 \cdot 10^{-23}$, $\hbar = 1.054 \cdot 10^{-34}$, $T = 2.728$, we find:

$$4.1281 \cdot 10^8 m^{-3} = 4.1281 \cdot 10^5 \tag{5}$$

photons per litre.

---

[1] See, for example, Planck's law on Wikipedia

[2] We can find this as follows: the number of (field) modes with magnitude of their wavevector between k and $k + dk$ is $4\pi k^2 dk$ divided by the volume $(\pi/L)^3$. But this is for a complete spherical shell; we're only interested in one octant. Therefore we multiply with $\frac{1}{8}$. Finally, we have to take both polarizations into account: an extra factor of 2. Result: $\rho(k)dk = k^2 dk/\pi^2$. Convert this by a simple substition of $\omega = ck$. Result: $\rho(\omega)d\omega = \omega^2 d\omega/\pi^2 c^3$.